

On runoff generation and the distribution of storage deficits

APPENDIX A

Appendix A presents the formulation of the analytical PDM-BETA5 soil moisture module. The model can be represented by Figure A.1. The river catchment is considered to be a continuous distribution of storage elements, designated as the store, as is the case in the PDM.

In Figure A.1 the store full capacity line is a smooth linear line and storage elements have depths between 0 and C_{max} . C_{max} is a parameter for the maximum store capacity depth. As in the case of the BETA3 and BETA4 this parameter can be fixed which allows the beta function to determine the minimum, maximum and distribution of the store depths which reduces the number of free parameters.

The module requires a calculation of the storage S_t at every timestep. A starting soil moisture storage is given to the model and thereafter the storage S_t is calculated at each timestep t . As shown in Figure A.1, at some arbitrary time t the soil moisture in the store is below total full storage capacity so that some elements of the store are full and some are below full capacity. x_{crit} represents the distance in the

beta function for which the store elements are full as shown and the water level is horizontal.

In Figure A.1 the store zero capacity level is a convenient datum from which to measure water depths. Then, using x as a distance of measure along the range of storage elements, with range from 0 to 1, the available water storage element at x above the zero flow level is given by $y_{c,x}$ and the storage at time t is denoted by $y_{s,x,t}$ or $y_{crit,x,t}$ depending on location. For an element dx of storage from 0 to x_{crit} the storage depth of the element is given by

$$y_{s,x,t} = C_{max}x \quad (A.1)$$

and for stores from x_{crit} to 1 the storage depth of the element is given by

$$y_{crit,x,t} = C_{max}x_{crit} \quad (A.2)$$

and the incremental volume of storage at point x is given by

$$S_{x,t} = A_x \cdot y_{s,x,t} \quad \text{for } x = 0 \quad \text{to} \quad x = x_{crit} \quad (A.3)$$

$$S_{x,t} = A_x \cdot y_{s,x,t} \quad \text{for } x = x_{crit} \quad \text{to} \quad x = 1 \quad (A.4)$$

where A_x is the area of the storage element dx at x which also depends on the probability density of the point dx so that

$$A_x = f(x)dx \quad (A.5)$$

At the point x the probability of the storage element dx of capacity $y_{c,x}$ above the datum is given by the beta function with parameters z and w , which is

$$beta(z, w) = \int_0^1 x^{z-1} (1-x)^{w-1} dx \quad (A.6)$$

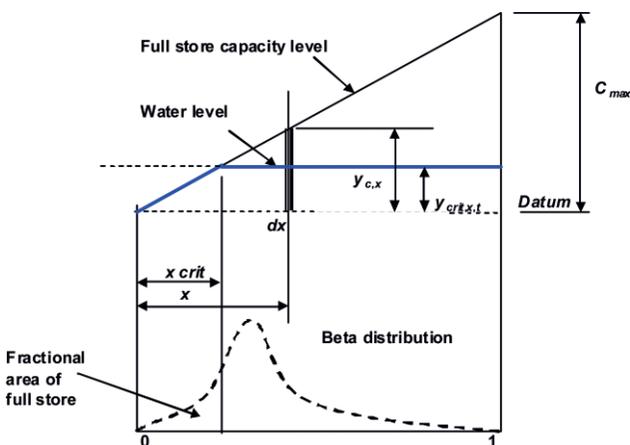


Figure A.1 | PDM-BETA5 soil moisture module.

so that in its probability density form this becomes

$$f(x) = \frac{x^{z-1}(1-x)^{w-1}}{\text{beta}(z, w)} \quad (\text{A.7})$$

The average storage depth across the whole catchment, S_t at time t , is then given by

$$S_t = \int_0^{x_{crit}} y_{s,x,t} f(x) dx + \int_{x_{crit}}^1 y_{crit,x,t} f(x) dx \quad (\text{A.8})$$

and substituting Equations (A.1), (A.2) and (A.6) into (A.8) gives

$$S_t = \frac{C_{max}}{\text{beta}(z, w)} \int_0^{x_{crit}} x^z (1-x)^{w-1} dx + \frac{C_{max} \cdot x_{crit}}{\text{beta}(z, w)} \times \int_{x_{crit}}^1 x^{z-1} (1-x)^{w-1} dx \quad (\text{A.9})$$

The incomplete beta function (for example *betainc* in the Matlab routine) is given by the normalised form of the beta function so that

$$\text{betainc}(x, z, w) = \frac{1}{\text{beta}(z, w)} \int_0^x x^{z-1} (1-x)^{w-1} dx \quad (\text{A.10})$$

From this

$$\text{beta}(z+1, w) \cdot \text{betainc}(x, z+1, w) = \int_0^x x^z (1-x)^{w-1} dx \quad (\text{A.11})$$

Also,

$$\text{beta}(z, w)(1 - \text{betainc}(x, z, w)) = \int_x^1 x^{z-1} (1-x)^{w-1} dx \quad (\text{A.12})$$

Substituting Equations (A.11) and (A.12) into (A.9) gives after rearranging,

$$S_t = \frac{C_{max}}{\text{beta}(z, w)} \text{beta}(z+1, w) \cdot \text{betainc}(x_{crit}, z+1, w) + C_{max} x_{crit} (1 - \text{betainc}(x_{crit}, z, w)) \quad (\text{A.13})$$

The solution for S_t in Equation (A.13) therefore requires the distance x_{crit} . Or, for a given storage, the calculation of x_{crit} can then be found by minimising

$$0 = \frac{C_{max}}{\text{beta}(z, w)} \text{beta}(z+1, w) \cdot \text{betainc}(x_{crit}, z+1, w) + C_{max} x_{crit} (1 - \text{betainc}(x_{crit}, z, w)) - S_t \quad (\text{A.14})$$

by using the Matlab *fzero* function or some other similar routine. However, this minimisation function is a slow computation.

An alternative method for the solution of x_{crit} , which may work faster in some computer routines, is to generate a look up table for values of S_t based on

$$S_t = \frac{C_{max}}{\text{beta}(z, w)} \text{beta}(z+1, w) \cdot \text{betainc}(x, z+1, w) + C_{max} x (1 - \text{betainc}(x, z, w)) \quad (\text{A.15})$$

As an example, which was found to give acceptable PDM-BETA5 model accuracy, 10,000 points for S_t were generated with x ranging from 0 to 1, i.e., x input increasing by steps of 0.0001.

Drainage, evaporation and rainfall

In timestep t drying takes place to lower the water level from the dashed line to the new water level shown by the dotted line. The drainage $q_{s,t}$ from the store is a linear or non-linear function of the storage S_t so that

$$q_{s,t} = f(S_t) \quad (\text{A.16})$$

The actual evaporation AE_t from the store was applied using the same function used in the PDM (Senbeta *et al.* 1999):

$$AE_t = \left[1 - \frac{S_{\max} - S_t}{S_{\max}} \right]^\alpha PE_t \quad (A.17)$$

where PE_t is the potential evaporation at time t and α is an evaporation constant that commonly takes a value of 2. S_{\max} is the maximum mean storage capacity of the river catchment that is obtained from Equation (A.15) and by taking x as 1.

If Rf_t is the rainfall at time t , the effective rainfall Rfe_t is

$$Rfe_t = Rf_t - q_t - AE_t \quad (A.18)$$

A drying situation is determined by a negative value of Rfe_t . A positive value of Rfe_t is a wetting situation.

Drying situation

Diagrammatically the drying situation is shown in Figure A.2.

The new water level is determined by drainage from the soil moisture store and evaporation which is greater than the rainfall. If S_t is the storage depth deficit at the start of the

timestep, then the new storage S_{t+1} becomes

$$S_{t+1} = S_t + Rfe_t \quad (A.19)$$

From this the critical distance x_{crit} is determined from the lookup table generated by Equation (A.15).

Wetting situation

Diagrammatically the wetting situation is shown in Figure A.3.

In this case the new water level $y_{\text{crit},t+1}$ will be determined by the effective precipitation so that

$$y_{\text{crit},t+1} = y_{\text{crit},t} + Rfe_t \quad (A.20)$$

Using Equation (A.2), x_{crit} and the new soil moisture storage S_{t+1} can be calculated.

In the wetting situation there is filling of some storage elements and a potential overflow from some of the store elements depending on the beta distribution of all the storage elements and the starting water level. The volume of overflow can also be expressed as an average depth across the catchment so that

$$V_{o,t} = Rfe_t - (S_{t+1} - S_t) \quad \text{if } V_{o,t} > 1 \quad (A.21)$$

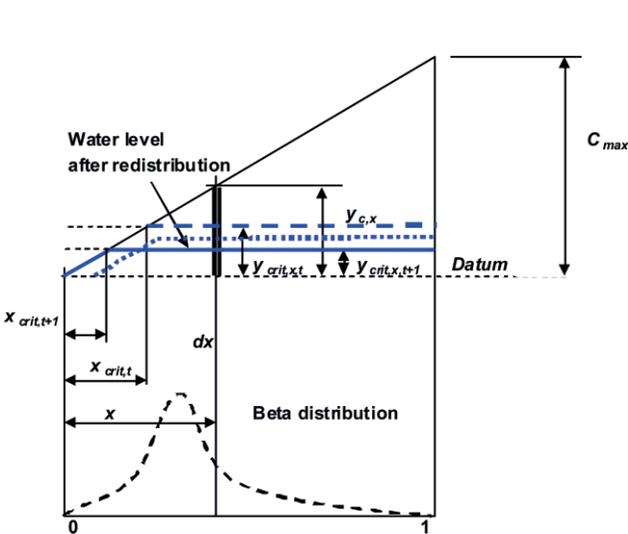


Figure A.2 | PDM-BETA5 drying.

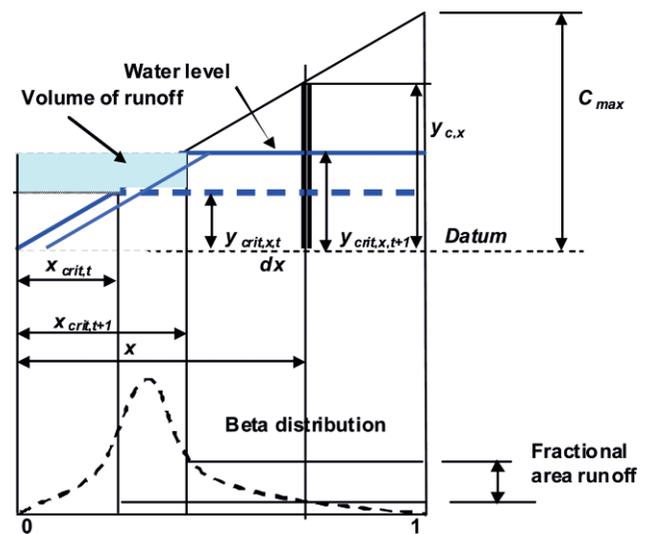


Figure A.3 | PDM-BETA5 module wetting.

or

$$V_{o,t} = 0 \quad \text{if } V_{o,t} < 1 \quad (\text{A.22})$$

REFERENCE

Senbeta, D. A., Shamseldin, A. Y. & O'Connor, K. M. 1999
Modification of the probability-distributed interacting storage
capacity model. *J. Hydrol.* **224**, 149–168.