A well-balanced spectral volume scheme with the wetting–drying property for the shallow-water equations

APPENDIX: PROOF OF THEOREM 1

From the description of the SV method, it is clear that it reduces to a finite-volume method, where variables are reconstructed efficiently by grouping the FV into spectral cells. Following this observation, we simplify the notation, dropping the subscripts \( i \) and \( j \), which refer to the SVs and the FV contained therein, and use a unique subscript \( l \), which refers to the generic FV of the domain discretization. In the present case, the subscript \( l \) can be defined as \( l = \frac{i - 1}{j} \), with \( i = 1, 2, \ldots, N \) and \( j = 1, 2, 5 \): of course, \( l \) ranges between 1 and 3 NS.

From inspection of Equation (8) we observe that the equation for the evolution in time of the water depth \( h \) can then be expressed, using the Euler algorithm for the time discretization, as

\[
\begin{align*}
\frac{\Delta t}{\Delta x_l} f^b(\tilde{u}_{l,1/2}^+, \tilde{u}_{l,1/2}^-) - f^b(\tilde{v}_{l-1/2}^+, \tilde{v}_{l-1/2}^-) &= \frac{\Delta t}{\Delta x_l} \left[ \tilde{h}_{l+1} - \tilde{h}_l \right], \\
l &= 1, 2, \ldots, k \text{ NS} & (A1)
\end{align*}
\]

and \( k = 3 \) is the number of FV in each SV. Formally, Equation (A1) is also representative of the case that there is no high-order reconstruction of variables: this scheme is only first-order accurate in space, and \( k = 1 \) must be assumed in Equation (A1).

Now, it is convenient to split the proof of Theorem 1 into simpler Lemmas. Regarding the C-property, the following proposition can be proved.

**Lemma 1.** The third-order SV Scheme (8) for the numerical approximation of the one-dimensional shallow-water Equation (1) satisfies the C-property exactly, also in the case of the emerging bottom.

**Proof.** Hypothesize that the water is initially at rest, having \( U = 0 \) and \( \xi = \text{constant everywhere} \). First consider the case in which \( C_{i-1}, C_i \) and \( C_{i+1} \) are wet. Reconstruction of variables supplies \( hU = 0 \) and \( \xi = \text{constant at interfaces between FV} \) and at quadrature points in the cells. It is easy to verify that \( \mathbf{s}_{li} = 0 \) and \( \tilde{u}_{l+1/2}^- = \tilde{u}_{l+1/2}^- \). In particular it is apparent that \( \tilde{h}_{i+1/2}^- = \tilde{h}_{i+1/2}^- \) and \( \tilde{h}_{i-1/2}^- = \tilde{h}_{i-1/2}^- \). Then it follows that

\[
\mathbf{f}_{l+1/2}^- = \mathbf{f}_{\text{HLL}}(\tilde{u}_{l+1/2}^-, \tilde{u}_{l+1/2}^-) = \mathbf{f}(\tilde{u}_{l+1/2}^-) = \frac{g}{2} \left[ \frac{0}{\hat{h}_{i+1/2}^-} \right].
\]  

(A2)

Being that \( \xi \) is constant in the whole numerical domain, Equation (23) yields

\[
\mathbf{s}_l = \frac{g}{\Delta x_l} \left[ \frac{0}{\hat{h}_{i+1/2}^-} \right].
\]  

(A3)

After substitution in Equation (8) we obtain

\[
\frac{\Delta u}{\Delta t} = 0
\]  

(A4)

and then the water remains at rest. Significantly, limitation of variables does not alter the C-property, in that, if water is initially at rest, the application of the limitation procedure yields again \( hU = 0 \) and \( \xi = \text{constant everywhere} \).

If \( C_{i-1} \) and \( C_i \) are wet, but \( C_{i+1} \) is dry, with \( z_{i+1} > \xi \), the reconstruction described ensures that \( \tilde{h}_{i+1/2}^- = 0 \) and \( \tilde{h}_{i-1/2}^- = 0 \): again one has \( \tilde{h}_{i+1/2}^- = \tilde{h}_{i+1/2}^- \) and \( \tilde{h}_{i-1/2}^- = \tilde{h}_{i-1/2}^- \), and the proof is the same as in the preceding case.

The preservation of non-negativity of water depth \( h \) by interface is sufficient to ensure the preservation of non-negativity in finite-volume first-order schemes: we first recall the following Lemma, which contains some result by Audusse et al. (2004) and Bouchut (2004).

**Lemma 2.** Assume that the homogeneous numerical flux \( \mathbf{f}_{\text{HLL}}(u_l, u_0) \) preserves the non-negativity of the depth \( h \) by
interface with a numerical speed \( \sigma(u_L, u_R) \geq 0 \), which means that whenever the CFL condition

\[
\sigma(u_L, u_R) \Delta t \leq \min \{ \Delta x_L, \Delta x_R \} \tag{A5}
\]

holds, together with the conditions \( h_L \geq 0 \) and \( h_R \geq 0 \), we have

\[
\begin{align*}
 h_L - \frac{\Delta t}{\Delta x_L} [f^h(u_L, u_R) - h_L U_L] & \geq 0 \\
 h_R - \frac{\Delta t}{\Delta x_R} [h_R U_R - f^h(u_L, u_R)] & \geq 0 .
\end{align*}
\tag{A6}
\]

Then, if the following CFL condition

\[
\max \{ \sigma(\tilde{u}_{i-1/2}, \tilde{u}_{i+1/2}), \sigma(\tilde{u}_{i+1/2}, \tilde{u}_{i+1/2}) \} \frac{\Delta t}{\Delta x_i} \leq \frac{1}{2} \ \forall i 
\tag{A7}
\]

is satisfied, the homogeneous numerical flux \( f_{\text{HLL}}(u_L, u_R) \) preserves the non-negativity of the finite-volume scheme (A1) with \( k = 1 \), that is

\[
h_i^n \geq 0 \ \forall i \Rightarrow h_i^{n+1} \geq 0 \ \forall i .
\tag{A8}
\]

**Proof.** The proof follows those of Proposition 2.5 in Bouchut (2004) and of Proposition 2.2 in Audusse et al. (2001), and is not repeated here. As is customary, in Equations (A5) and (A6) the subscripts \( L \) and \( R \) refer to the quantities on the left and on the right of the interface, respectively.

**Remark 1.** The HLLE approximate Riemann solver, used throughout this paper, preserves non-negativity by interface if we chose \( \sigma = \max( -\alpha_L, \alpha_R ) \), where \( \alpha_L \) and \( \alpha_R \) are the signal speeds by Einfeldt (1988): in fact, the cited choice of signal speeds ensures the non-negativity of the depth in the middle state of the HLL approximate solution of the Riemann problem (Harten et al. 1983).

From Lemma 2, it is clear that HLLE preserves non-negativity of the first-order finite-volume scheme, under the half CFL condition, also when the hydrostatic reconstruction is applied.

Now, the following Lemma, modified from Perthame & Shu (1996), can be easily demonstrated.

**Lemma 3.** If the homogeneous numerical flux \( f_{\text{HLL}}(u_L, u_R) \) preserves the non-negativity of the depth \( h \) in the first-order finite-volume scheme, under the CFL condition (A7), then it preserves the non-negativity of the third-order scheme (8) under the reduced CFL condition

\[
\frac{\Delta t}{\Delta x_i} \max \{ \sigma(\tilde{u}_{i-1/2}, \tilde{u}_{i+1/2}), \sigma(\tilde{u}_{i+1/2}, \tilde{u}_{i+1/2}) \} \leq \frac{1}{12} \ \forall i .
\tag{A9}
\]

**Proof.** If we set

\[
u_i^n = v_i + \alpha(\tilde{u}_{i+1/2} + \tilde{u}_{i-1/2})
\tag{A10}
\]

and regard Equation (A1) as a third-order accurate FV scheme, we can split it into three distinct first-order schemes:

\[
\begin{align*}
\dot{h}_i^+ &= \dot{h}_{i+1/2} - \frac{\Delta t}{\Delta x_i} [f^h(\tilde{u}_{i+1/2}, \tilde{u}_{i+1/2}) - f^h(v_i, \tilde{u}_{i+1/2})] \\
\dot{\lambda}_i &= \lambda_i - \frac{\Delta t}{\Delta x_i} [f^h(v_i, \tilde{u}_{i+1/2}) - f^h(\tilde{u}_{i+1/2}, \tilde{u}_{i+1/2})] \\
\dot{h}_i^- &= \dot{h}_{i-1/2} - \frac{\Delta t}{\Delta x_i} [f^h(\tilde{u}_{i-1/2}, v_i) - f^h(\tilde{u}_{i-1/2}, \tilde{u}_{i+1/2})]
\end{align*}
\tag{A11}
\]

where the superscripts referring to the time level have been dropped for ease of notation, and \( \lambda_i \) represents the first component of \( v_i \) The chosen splitting satisfies the congruency conditions

\[
\lambda_i = h_i^n - \alpha(\tilde{h}_{i+1/2} + \tilde{h}_{i-1/2}), \quad \tilde{\lambda}_i = h_i^{n+1} - \alpha(\tilde{h}_{i+1} + \tilde{h}_i) .
\tag{A12}
\]

Now, we observe that the reconstruction of \( h \) is quadratic in the generic FV, and then

\[
h_i^n = \frac{\tilde{h}_{i+1/2} + 4h_C + \tilde{h}_{i-1/2}}{6} \geq \frac{\tilde{h}_{i+1/2} + 4h_{i+1}^\text{min} + \tilde{h}_{i-1/2}}{6},
\tag{A13}
\]

where \( h_C \) is the water depth evaluated at the centre of the FV, while \( h_{i+1}^\text{min} \) is the minimum value of \( h \) in the same FV. Since the limitation rules ensure that \( \tilde{h}_{i+1/2} \geq \tilde{h}_{i-1/2} \geq 0 \), \( h_{i+1/2} \geq \tilde{h}_{i+1/2} \geq 0 \), and \( h_{i+1}^{\text{min}} \geq 0 \), then \( h_i^n \geq (\tilde{h}_{i+1/2} + \tilde{h}_{i-1/2})/6 \), and the non-negativity of \( \lambda_i \) is satisfied by \( \alpha = 1/6 \).

Finally, the quantities \( \tilde{h}_i^+ \), \( \tilde{\lambda}_i \) and \( \tilde{\lambda}_i \) are separately non-negative if the CFL condition (A7) is met for each equation of the system (A11), and then Equation (A9) follows easily.

In conclusion, Lemma 1 states that the third-order SV Scheme (8) preserves the C-property, while Lemma 3 ensures that the same scheme preserves the non-negativity under a
properly reduced CFL condition, when the Euler algorithm is used for the march in time. The proof of Theorem 1 is complete if we observe that the optimal three-stage, third-order TVD Runge-Kutta time discretization consists of a convex combination of Euler forward steps (Kurganov & Levy 2002).

REFERENCES


