Experimental comparison of canal models for control purposes using simulation and laboratory experiments

APPENDIX A

The Integrator Delay and Integrator Delay Zero models

Both models are based on the linearized SV equations. The SV equations can be linearized around a steady state regime (noted with an underscore zero). The steady state water depth can be expressed as

\[ h = H - H_0, \]

(A1)

and the discharge as

\[ q = Q - Q_0. \]

(A2)

Then the linearized equations are

\[ T_{bw}(x) \frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0, \]

(A3)

\[ \frac{\partial q}{\partial t} + 2V_0(x) \frac{\partial q}{\partial x} - \beta_0(x)q + \left( C_0(x)^2 - V_0(x)^2 \right) \]

\[ T_{bw}(x) \frac{\partial h}{\partial x} - \gamma_0(x) = 0, \]

(A4)

where \( A_0 \) is the wetted area, \( V_0 \) is the velocity, \( S_b \) is the bed slope, \( g \) is the acceleration of gravity and \( \beta \) and \( \gamma \) are parameters and can be expressed as

\[ \beta_0 = -\frac{2g}{V_0} \left( S_b - \frac{\partial H_0}{\partial x} \right), \]

(A5)

\[ \gamma_0 = V_0^2 \frac{\partial T_{bw}}{\partial x} + gT_{bw} \left( 1 + \kappa_0 \right) S_b - \left( 1 + \kappa_0 - F_0^2(\kappa_0 - 2) \right) \frac{\partial H_0}{\partial x}. \]

(A6)

The Integrator Delay model

The construction of the ID model is adopted from Schuurmans (1995). In order to construct the integrator delay model, the Laplace transform of the linearized transfer functions (A3) and (A4) can be expressed in a matrix form as

\[ \begin{pmatrix} h(x, s) \\ q(x, s) \end{pmatrix} = c_1 \left( -\frac{\alpha_1(s)}{T_{bw}} \right) e^{\alpha_1(s)x} + c_2 \left( -\frac{\alpha_2(s)}{T_{bw}} \right) e^{\alpha_2(s)x}, \]

(A7)

where \( c_1 \) and \( c_2 \) are the integration constants and \( \alpha_1 \) and \( \alpha_2 \) are the roots of the characteristic equation associated with (A3) and (A4). For each canal pool, the boundary conditions can be defined in terms of flow rates, hence the integration coefficients of (A7) can be written as

\[ c_1 = \frac{q(L, s) - e^{\alpha_1 L}q(0, s)}{e^{\alpha_1 L} - e^{\alpha_2 L}}, \]

(A8)

\[ c_2 = \frac{q(L, s) - e^{\alpha_2 L}q(0, s)}{e^{\alpha_1 L} - e^{\alpha_2 L}}. \]

(A9)

In the backwater part the waves hardly deform, and the dynamics can be approximated by low frequency approximation by considering \( s \to 0 \) from (A7)–(A9):

\[ h(x, s) = \frac{K_1}{s} q(L - L_e, s) - \frac{K_2}{s} q(L, s), \]

(A10)

where \( L_e \) is the length of the backwater part and the coefficients \( K_1 \) and \( K_2 \) are expressed as

\[ K_1 = \frac{\gamma_0}{T_{bw}^2 \left( c^2 - V_0^2 \right)^2 \left( 1 - e^{-\gamma_0 T_{bw} \left( c^2 - V_0^2 \right) L_e} \right)}, \]

(A11)

\[ K_2 = K_1 e^{-\gamma_0 T_{bw} \left( c^2 - V_0^2 \right) L_e}. \]

(A12)
The approximation of (A10) can be simplified: as the water profile is close to the horizontal in the backwater part, \( \gamma_0 \) approaches zero, parameters \( K_1 \) and \( K_2 \) can be approximated as

\[
\lim_{\gamma_0 \to 0} K_1 = \frac{\gamma_0}{T_0^2 (e^2 - V_0^2) \left( 1 - e^{-|\gamma_0/\gamma_0(e^2-V_0^2)|} \right)} = \frac{1}{T_0 L_e}. \tag{A13}
\]

Therefore in the backwater part (A10) can be written as

\[
h(x, s) = \frac{1}{A_0 s} q(L - L_e, s) - q(L, s). \tag{A14}
\]

To approximate the uniform part the kinematic wave equation can be written in the form of a transfer function as

\[
e^{2sx} \approx e^{-(2x/[1+c_0 V_0])s}, \tag{A15}
\]

where the parameter \( c_0 \) can be obtained as

\[
c_0 = 1 + \frac{4P_0}{3 T_0} \frac{d R}{d h_0}, \tag{A16}
\]

where \( P_0 = A_0/R_0 \) is the wet perimeter. By combining the two approximations (A15) and (A16), Equation (19) can be obtained.

### The Integrator Delay Zero model

The IDZ model gives an approximation for the integrator gain and the delay (as the ID model) and extends it with a zero at high frequencies (Litrico & Fromion 2004c). The model can be computed analytically. First, the canal pool is divided into two parts: the downstream backwater part and the upstream uniform part. For both parts, low and high frequency approximation is calculated: for low frequencies the dominating parameter is the integrator. At high frequencies the gravity waves are dominant; for simplicity these waves are approximated with constant gain. The development of the model is explained in detail in Litrico & Fromion (2004c). Here the given explanation is summarized. It involves the Laplace transform of the linearized SV

Equations (A3) and (A4) in a form of

\[
\frac{d}{ds} \left( \begin{array}{c} q(x, s) \\ h(x, s) \end{array} \right) = \theta(x) \left( \begin{array}{c} q(x, s) \\ h(x, s) \end{array} \right), \tag{A17}
\]

where \( \theta \) is the transfer matrix

\[
\theta(x) = \left( \begin{array}{cc} 0 & -T_0(x) s \\ -s + \beta_0(x) & 2V_0(x)T_0(x) s + \gamma_0(x) \end{array} \right). \tag{A18}
\]

(A17) can be ordered in a way that the water levels can be expressed in the form

\[
\left( \begin{array}{c} h(0, s) \\ h(x, s) \end{array} \right) = \left( \begin{array}{cc} p_{11}(X, s) & p_{12}(X, s) \\ p_{21}(X, s) & p_{22}(X, s) \end{array} \right) \left( \begin{array}{c} q(0, s) \\ q(X, s) \end{array} \right). \tag{A19}
\]

In steady state case, the transfer matrix can be assumed to be constant and each entry can be approximated using the geometry of the canal and the flow conditions. The approximations are presented in the following form:

\[
p_{11} = \frac{1}{A_0 s} + \tilde{p}_{11}, \tag{A20}
\]

\[
p_{12} = -\left( \frac{1}{A_0 s} + \tilde{p}_{12} \right) e^{-h_0 s}, \tag{A21}
\]

\[
p_{21} = -\left( \frac{1}{A_0 s} + \tilde{p}_{21} \right) e^{-2h_0 s}, \tag{A22}
\]

\[
p_{22} = -\frac{1}{A_0 s} - \tilde{p}_{22}. \tag{A23}
\]

The derivation of the parameters in Equations (A20)–(A23) are explained in Litrico & Fromion (2004c). As the downstream water level is to be controlled, it can be expressed using the first row of (A21):

\[
h(s) = \tilde{p}_{21}q_{\text{in}}(s) + \tilde{p}_{22}q_{\text{out}}(s). \tag{A24}
\]
APPENDIX B

Controller tuning

The matrix $R$ contains the corresponding weights to the input, the matrix $P$ contains the weights on the state. In this case the input is the change in discharge, therefore matrix $R$ penalizes the changes in discharge. Matrix $P$ penalizes the state. The current water level error $e_i(k)$ was chosen to be penalized, therefore only the diagonal elements of $P$ corresponding to the current water level errors in the state are non-zero. The weights on the water level error and change in discharge are chosen using Bryson’s rule (Bryson & Ho 1969): the weights are the reciprocals of the squares of the maximum allowed values. For example, the entries of matrix $P$ can be calculated as

$$ p_i = \frac{1}{e_{MAVE}^2}, \quad (B1) $$

where $e_{MAVE}$ is the maximum allowed water level error. $e_{MAVE}$ is chosen to be 3 cm and 5 cm for the Corning and the laboratory canal, respectively. The penalty on the discharge change is 0.1 and 0.01 m$^3$/s for the Corning and the laboratory canal, respectively.

The prediction horizon was chosen to be 20 and 17 control time steps, respectively, which corresponds to about prediction ahead of 3 hours for the Corning canal and 3 minutes for the laboratory canal.

REFERENCES

